

String Submodular Functions with Curvature Constraints

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Abstract—The problem of objectively choosing a string of actions to optimize an objective function that is *string submodular* has been considered in [1]. There it is shown that the greedy strategy, consisting of a string of actions that only locally maximizes the step-wise gain in the objective function, achieves at least a $(1 - e^{-1})$ -approximation to the optimal strategy. This paper improves this approximation by introducing additional constraints on curvatures, namely, *total backward curvature*, *total forward curvature*, and *elemental forward curvature*. We show that if the objective function has total backward curvature σ , then the greedy strategy achieves at least a $\frac{1}{\sigma}(1 - e^{-\sigma})$ -approximation of the optimal strategy. If the objective function has total forward curvature ϵ , then the greedy strategy achieves at least a $(1 - \epsilon)$ -approximation of the optimal strategy. Moreover, we consider a generalization of the diminishing-return property by defining the elemental forward curvature. We also consider the problem of maximizing the objective function subject to a *string-matroid* constraint. We investigate two applications of string submodular functions with curvature constraints.

I. INTRODUCTION

A. Background

We consider the problem of optimally choosing a string of actions to maximize an objective function over a finite horizon. Let \mathbb{A} be a set of possible actions. At each stage i , we choose an action a_i from \mathbb{A} . We use $A = (a_1, a_2, \dots, a_k)$ to denote a string of actions taken over k consecutive stages, where $a_i \in \mathbb{A}$ for $i = 1, 2, \dots, k$. We use \mathbb{A}^* to denote the set of all possible strings of actions (of arbitrary length, including the empty string). Let $f : \mathbb{A}^* \rightarrow \mathbb{R}$ be an objective function, where \mathbb{R} denotes the real numbers. We wish to find a string $M \in \mathbb{A}^*$, with a length $|M|$ not larger than K , to maximize the objective function:

$$\begin{aligned} & \text{maximize } f(M) \\ & \text{subject to } M \in \mathbb{A}^*, |M| \leq K. \end{aligned} \quad (1)$$

The solution to (1), which we call the *optimal strategy*, can be found using dynamic programming (see, e.g. [2]). More specifically, this solution can be expressed with *Bellman's equations*. However, the computational complexity of finding an optimal strategy grows exponentially with respect to the

size of \mathbb{A} and the length constraint K . On the other hand, the greedy strategy, though suboptimal in general, is easy to compute because at each stage, we only have to find an action to maximize the step-wise gain in the objective function. The question we are interested in is: How good is the greedy strategy compared to the optimal strategy in terms of the objective function? This question has attracted widespread interest, which we will review in the next section.

In this paper, we extend the concept of set submodularity in combinatorial optimization to bound the performance of the greedy strategy with respect to that of the optimal strategy. Moreover, we will introduce additional constraints on curvatures, namely, total backward and forward curvatures and elemental forward curvature, such that the greedy strategy achieves a good approximation of the optimal strategy. We will investigate the relationship between the approximation bounds for the greedy strategy and the curvature constraints. These results have many potential applications in closed-loop control problems such as portfolio management (see, e.g. [3]), closed-loop sensor management (see, e.g., [4]), and influence in social networks (see, e.g., [5]).

B. Related Work

Submodular set functions play an important role in combinatorial optimization. Let X be a ground set and $g : 2^X \rightarrow \mathbb{R}$ be an objective function defined on the power set 2^X of X . Let \mathcal{I} be a non-empty set of subsets of X . Suppose that \mathcal{I} has the *hereditary* and *augmentation* properties. Then, we call (X, \mathcal{I}) a matroid [6]. The goal is to find a set in \mathcal{I} to maximize the objective function:

$$\begin{aligned} & \text{maximize } g(N) \\ & \text{subject to } N \in \mathcal{I}. \end{aligned} \quad (2)$$

Suppose that $\mathcal{I} = \{S \subset X : \text{card}(S) \leq k\}$ for a given k , where $\text{card}(S)$ denotes the cardinality of S . Then, we call (X, \mathcal{I}) a *uniform matroid*.

The main difference between (1) and (2) is that the objective function in (1) depends on the order of elements in the string M , while the objective function in (2) is independent of the order of elements in the set N . To further explain the difference, we use $\mathcal{P}(M)$ to denote a permutation of a string M . Note that for M with length k , there exists $k!$ permutations. In (1), suppose that for any $M \in \mathbb{A}^*$ we have $f(M) = f(\mathcal{P}(M))$ for any \mathcal{P} . Then problem (1) reduces to problem (2). In other words, we can view the second problem as a special case of the first problem. Moreover, there can be

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repeated identical elements in a string, while a set does not contain identical elements (we note that this difference can be bridged by allowing multisets).

Finding the solution to (2) is NP-hard and a tractable alternative is to use a greedy algorithm. The greedy algorithm starts with the empty set, and incrementally adds an element to the current solution giving the largest gain in the objective function. Theories for maximizing submodular set functions and their applications have been intensively studied in recent years [7]–[31]. The main idea is to compare the performance of the greedy algorithm with that of the optimal solution. Suppose that the set objective function g is non-decreasing: $g(A) \leq g(B)$ for all $A \subset B$; and $g(\emptyset) = 0$ where \emptyset denotes the empty set. Moreover, the function has the *diminishing-return property*: For all $A \subset B \subset X$ and $j \in X \setminus B$, we have $g(A \cup \{j\}) - g(A) \geq g(B \cup \{j\}) - g(B)$. Then, we say that g is a *submodular set function*. Nemhauser *et al.* [7] showed that the greedy algorithm achieves at least a $(1 - e^{-1})$ -approximation for (2) given that (X, \mathcal{I}) is a uniform matroid and the objective function is submodular. Fisher *et al.* [8] proved that the greedy algorithm provides at least a $1/2$ -approximation of the optimum for a general matroid. Conforti and Cornuéjols [9] showed that if the function g has a total curvature c , where

$$c = \max_{j \in X} \left\{ 1 - \frac{g(X) - g(X \setminus \{j\})}{g(\{j\}) - g(\emptyset)} \right\},$$

then the greedy algorithm achieves at least $\frac{1}{1+c}(1 - e^{-c})$ and $\frac{1}{1+c}$ -approximations of the optimal solution given that (X, \mathcal{I}) is a uniform matroid and a general matroid, respectively. Note that $c \in [0, 1]$, and if $c = 0$, then the greedy algorithm is the optimal solution; if $c = 1$, then the result is the same as that in [7]. Vondrák [10] showed that the *continuous greedy algorithm* achieves a $\frac{1}{c}(1 - e^{-c})$ -approximation for any matroid. Wang *et al.* [11]¹ provided approximation bounds in the case where the function has an elemental curvature constraint, which generalizes the notion of diminishing return.

Some recent papers [1],[12]–[14] have extended the notion of set submodularity to problem (1). Streeter and Golovin [12] showed that if the function f is *forward* and *backward* monotone: $f(M \oplus N) \geq f(M)$ and $f(M \oplus N) \geq f(N)$ for all $M, N \in \mathbb{A}^*$, where \oplus means string concatenation, and f has the diminishing-return property: $f(M \oplus (a)) - f(M) \geq f(N \oplus (a)) - f(N)$ for all $a \in \mathbb{A}$, $M, N \in \mathbb{A}^*$ such that M is a prefix of N , then the greedy strategy achieves at least a $(1 - e^{-1})$ -approximation of the optimal strategy. The notion of *string submodularity* and weaker sufficient conditions are established in [1] under which the greedy strategy still achieves at least a $(1 - e^{-1})$ -approximation of the optimal strategy. Golovin and Krause [14] introduced adaptive submodularity for solving stochastic optimization problems under partial observability.

C. Contributions

In this paper, we study the problem of maximizing submodular functions defined on strings. We impose additional

constraints on curvatures, namely, total backward and forward curvatures, and elemental forward curvature, which will be rigorously defined in Section II. The notion of total forward and backward curvatures is inspired by the work of Conforti and Cornuéjols [9]. However, the forward and backward algebraic structures are not exposed in the setting of set functions because the order of elements in a set does not matter. The notion of elemental forward curvature is inspired by the work of Wang *et al.* [11]. We have exposed the forward algebraic structure of this elemental curvature in the setting of string functions. Moreover, the result and technical approach in [11] are different from those in this paper. More specifically, the result in [11] uses the fact that the value of a set function evaluated at a given set does not change with respect to any permutation of this set. However, the value of a string function evaluated at a given string might change with respect to a permutation of this string. In Section III, we consider the maximization problem in the case where the strings are chosen from a uniform structure. Suppose that the string submodular function f has total backward curvature with respect to the optimal strategy $\sigma(O)$. Then, the greedy strategy achieves at least a $\frac{1}{\sigma(O)}(1 - e^{-\sigma(O)})$ -approximation of the optimal strategy. Suppose that the string submodular function f has total forward curvature ϵ . Then, the greedy strategy achieves at least a $(1 - \epsilon)$ -approximation of the optimal strategy. We also generalize the notion of diminishing return by defining elemental forward curvature η . The greedy strategy achieves at least a $1 - (1 - \frac{1}{K_\eta})^K$ -approximation, where $K_\eta = (1 - \eta^K)/(1 - \eta)$. In Section IV, we consider the maximization problem in the case where the strings are chosen from a non-uniform structure by introducing the notion of string-matroid. Suppose that the string submodular function f has total backward curvature with respect to the optimal strategy $\sigma(O)$. Then, the greedy strategy achieves at least a $1/(1 + \sigma(O))$ -approximation. We also provide the approximation bounds for the greedy strategy when the function has total forward curvature and elemental forward curvature. In Section V, we consider two applications of string submodular functions with curvature constraints.

II. STRING SUBMODULARITY, CURVATURE, AND STRATEGIES

A. String Submodularity

We now introduce notation (same as those in [1]) to define string submodularity. Consider a set \mathbb{A} of all possible actions. At each stage i , we choose an action a_i from \mathbb{A} . Let $A = (a_1, a_2, \dots, a_k)$ be a string of actions taken over k stages, where $a_i \in \mathbb{A}$, $i = 1, \dots, k$. Let the set of all possible strings of actions be

$$\mathbb{A}^* = \{(a_1, a_2, \dots, a_k) | k = 0, 1, \dots \text{ and } a_i \in \mathbb{A}, i = 1, \dots, k\}.$$

Note that $k = 0$ corresponds to the empty string (no action taken), denoted by \emptyset . For a given string $A = (a_1, a_2, \dots, a_k)$, we define its *string length* as k , denoted $|A| = k$. If $M = (a_1^m, a_2^m, \dots, a_{k_1}^m)$ and $N = (a_1^n, a_2^n, \dots, a_{k_2}^n)$ are two strings in \mathbb{A}^* , we say $M = N$ if $|M| = |N|$ and $a_i^m = a_i^n$ for each $i = 1, 2, \dots, |M|$. Moreover, we define string *concatenation*

¹We gratefully thank Zengfu Wang for sending us a preprint of [11].

as follows:

$$M \oplus N = (a_1^m, a_2^m, \dots, a_{k_1}^m, a_1^n, a_2^n, \dots, a_{k_2}^n).$$

If M and N are two strings in \mathbb{A}^* , we write $M \preceq N$ if we have $N = M \oplus L$, for some $L \in \mathbb{A}^*$. In other words, M is a *prefix* of N .

A function from strings to real numbers, $f : \mathbb{A}^* \rightarrow \mathbb{R}$, is *string submodular* if

i. f has the *forward-monotone* property, i.e.,

$$\forall M, N \in \mathbb{A}^*, \quad f(M \oplus N) \geq f(M).$$

ii. f has the *diminishing-return* property, i.e.,

$$\begin{aligned} \forall M \preceq N \in \mathbb{A}^*, \forall a \in \mathbb{A}, \\ f(M \oplus (a)) - f(M) \geq f(N \oplus (a)) - f(N). \end{aligned}$$

In the rest of the paper, we assume that $f(\emptyset) = 0$. Otherwise, we can replace f with the marginalized function $f - f(\emptyset)$. From the forward-monotone property, we know that $f(M) \geq 0$ for all $M \in \mathbb{A}^*$.

We first state an immediate result from the definition of string submodularity.

Lemma 1: Suppose that f is string submodular. Then, for any string $N = (n_1, n_2, \dots, n_{|N|})$, we have

$$f(N) \leq \sum_{i=1}^{|N|} f((n_i)).$$

The proof uses a mathematical induction argument and is omitted for brevity.

B. Curvature

We define the *total backward curvature* of f by

$$\sigma = \max_{a \in \mathbb{A}, M \in \mathbb{A}^*} \left\{ 1 - \frac{f((a) \oplus M) - f(M)}{f((a)) - f(\emptyset)} \right\}. \quad (3)$$

We define the total backward curvature of f with respect to string $M \in \mathbb{A}^*$ by

$$\sigma(M) = \max_{N \in \mathbb{A}^*, 0 < |N| \leq K} \left\{ 1 - \frac{f(N \oplus M) - f(M)}{f(N) - f(\emptyset)} \right\}, \quad (4)$$

where K is the length constraint in (1). We will discuss the values $\sigma(M)$ can take in the next section. Suppose that f is backward-monotone; i.e., $\forall M, N \in \mathbb{A}^*, f(M \oplus N) \geq f(N)$. Then, we have $\sigma \leq 1$ and f has total curvature at most σ with respect to any $M \in \mathbb{A}^*$; i.e., $\sigma(M) \leq \sigma \forall M \in \mathbb{A}^*$. This fact can be shown using a simple derivation: For any $N \in \mathbb{A}^*$, we have

$$\begin{aligned} f(N \oplus M) - f(M) = \\ \sum_{i=1}^{|N|} f((n_i, \dots, n_{|N|}) \oplus M) - f((n_{i+1}, \dots, n_{|N|}) \oplus M), \end{aligned}$$

where n_i represents the i th element of N . From the definition of total backward curvature and Lemma 1, we obtain

$$\begin{aligned} f(N \oplus M) - f(M) &\geq \sum_{i=1}^{|N|} (1 - \sigma) f(n_i) \\ &\geq (1 - \sigma) f(N). \end{aligned}$$

Symmetrically, we define the *total forward curvature* of f by

$$\epsilon = \max_{a \in \mathbb{A}, M \in \mathbb{A}^*} \left\{ 1 - \frac{f(M \oplus (a)) - f(M)}{f((a)) - f(\emptyset)} \right\}. \quad (5)$$

Moreover, we define the total forward curvature with respect to M by

$$\epsilon(M) = \max_{N \in \mathbb{A}^*, 0 < |N| \leq K} \left\{ 1 - \frac{f(M \oplus N) - f(M)}{f(N) - f(\emptyset)} \right\}. \quad (6)$$

If f is string submodular and has total forward curvature ϵ , then it has total forward curvature at most ϵ with respect to any $M \in \mathbb{A}^*$; i.e., $\epsilon(M) \leq \epsilon \forall M \in \mathbb{A}^*$. Moreover, for a string submodular function f , it is easy to see that for any M , we have $\epsilon(M) \leq \epsilon \leq 1$ because of the forward-monotone property and $\epsilon(M) \geq 0$ because of the diminishing-return property.

We define the *elemental forward curvature* of the string submodular function by

$$\eta = \max_{a_i, a_j \in \mathbb{A}, M \in \mathbb{A}^*} \frac{f(M \oplus (a_i) \oplus (a_j)) - f(M \oplus (a_i))}{f(M \oplus (a_j)) - f(M)}. \quad (7)$$

For a forward-monotone function, we have $\eta \geq 0$. Moreover, note that the diminishing-return property is equivalent to the condition $\eta \leq 1$.

C. Strategies

We will consider the following two strategies.

1) *Optimal strategy:* Consider the problem (1) of finding a string that maximizes f under the constraint that the string length is not larger than K . We call a solution of this problem an *optimal strategy* (a term we already have used repeatedly before). Note that because the function f is forward monotone, it suffices to just find the optimal strategy subject to the stronger constraint that the string length is equal to K . In other words, the optimal strategy must be a string with length K .

2) *Greedy strategy:* A string $G_k = (a_1^*, a_2^*, \dots, a_k^*)$ is called *greedy* if $\forall i = 1, 2, \dots, k$,

$$\begin{aligned} a_i^* &= \arg \max_{a_i \in \mathbb{A}} f((a_1^*, a_2^*, \dots, a_{i-1}^*, a_i)) \\ &\quad - f((a_1^*, a_2^*, \dots, a_{i-1}^*)). \end{aligned}$$

Notice that the greedy strategy only maximizes the step-wise gain in the objective function. In general, the greedy strategy (also called the greedy string) is not an optimal solution to (1). In this paper, we establish theorems which state that the greedy strategy achieves at least a factor of the performance of the optimal strategy, and therefore serves in some sense to *approximate* an optimal strategy.

III. UNIFORM STRUCTURE

Let I be the subset of \mathbb{A}^* with maximal string length K : $I = \{A \in \mathbb{A}^* : |A| \leq K\}$. We call I a *uniform structure*. Note that the way we define uniform structure is similar to the way we define independent sets of uniform matroids. We will investigate the case of non-uniform structure in the next section. Now (1) can be rewritten as

$$\begin{aligned} &\text{maximize } f(M) \\ &\text{subject to } M \in I. \end{aligned}$$

We first consider the relationship between the total curvatures and the approximation bounds for the greedy strategy.

Theorem 1: Consider a string submodular function f . Let O be a solution to (1). Then, any greedy string G_K satisfies

- (i) $f(G_K) \geq \frac{1}{\sigma(O)}(1 - e^{-\sigma(O)})f(O)$,
- (ii) $f(G_K) \geq (1 - \max_{i=1, \dots, K-1} \epsilon(G_i))f(O)$.

Proof: (i) For any $M \in \mathbb{A}^*$ and any $N = (a_1, a_2, \dots, a_{|N|}) \in \mathbb{A}^*$, we have

$$\begin{aligned} & f(M \oplus N) - f(M) \\ &= \sum_{i=1}^{|N|} (f(M \oplus (a_1, \dots, a_i)) - f(M \oplus (a_1, \dots, a_{i-1}))) \end{aligned}$$

Therefore, using the forward-monotone property, there exists an element $a_j \in \mathbb{A}$ such that

$$\begin{aligned} & f(M \oplus (a_1, \dots, a_j)) - f(M \oplus (a_1, \dots, a_{j-1})) \\ & \geq \frac{1}{|N|} (f(M \oplus N) - f(M)). \end{aligned}$$

Moreover, the diminishing-return property implies that

$$\begin{aligned} & f(M \oplus (a_j)) - f(M) \\ & \geq f(M \oplus (a_1, \dots, a_j)) - f(M \oplus (a_1, \dots, a_{j-1})) \\ & \geq \frac{1}{|N|} (f(M \oplus N) - f(M)). \end{aligned}$$

Now let us consider the optimization problem (1). Using the property of the greedy strategy and the above inequality (substitute $M = G_{i-1}$ and $N = O$), for each $i = 1, 2, \dots, K$ we have

$$\begin{aligned} & f(G_i) - f(G_{i-1}) \\ & \geq \frac{1}{K} (f(G_{i-1} \oplus O) - f(G_{i-1})) \\ & \geq \frac{1}{K} (f(O) - \sigma(O)f(G_{i-1})). \end{aligned}$$

Therefore, we have

$$\begin{aligned} f(G_K) & \geq \frac{1}{K} f(O) + \left(1 - \frac{\sigma(O)}{K}\right) f(G_{K-1}) \\ & = \frac{1}{K} f(O) \sum_{i=0}^{K-1} \left(1 - \frac{\sigma(O)}{K}\right)^i \\ & = \frac{1}{\sigma(O)} \left(1 - \left(1 - \frac{\sigma(O)}{K}\right)^K\right) f(O). \end{aligned}$$

Note that

$$\frac{1}{\sigma(O)} \left(1 - \left(1 - \frac{\sigma(O)}{K}\right)^K\right) \rightarrow \frac{1}{\sigma(O)} (1 - e^{-\sigma(O)})$$

from above as $K \rightarrow \infty$. In consequence, we obtain the desired result.

(ii) Using a similar argument to that in (i), we have

$$\begin{aligned} & f(G_i) - f(G_{i-1}) \\ & \geq \frac{1}{K} (f(G_{i-1} \oplus O) - f(G_{i-1})) \\ & \geq \frac{1}{K} (f(G_{i-1}) + (1 - \epsilon(G_{i-1}))f(O) - f(G_{i-1})) \\ & = \frac{1}{K} (1 - \epsilon(G_{i-1}))f(O). \end{aligned}$$

Therefore, by recursion we have

$$\begin{aligned} f(G_K) &= \sum_{i=1}^K (f(G_i) - f(G_{i-1})) \\ &\geq \sum_{i=1}^K \frac{1}{K} (1 - \epsilon(G_{i-1}))f(O) \\ &\geq \frac{1}{K} K (1 - \max_{i=1, \dots, K-1} \epsilon(G_i))f(O) \\ &= (1 - \max_{i=1, \dots, K-1} \epsilon(G_i))f(O). \end{aligned}$$

■

Under the framework of maximizing submodular set functions, similar results are reported in [9]. However, the forward and backward algebraic structures are not exposed because the total curvature in [9] does not depend on the order of the elements in a set. In the setting of maximizing string submodular functions, the forward and backward algebraic structures are illustrated in the above theorem. To explain further, let us state the results in a symmetric fashion. Suppose that the diminishing-return property is stated in the backward way: $f((a) \oplus M) - f(M) \geq f((a) \oplus N) - f(N)$ for all $a \in \mathbb{A}$ and $M, N \in \mathbb{A}^*$ such that $N = (a_1, \dots, a_k) \oplus M$. Moreover, a string $\hat{G}_k = (a_1^*, a_2^*, \dots, a_k^*)$ is called *backward-greedy* if

$$\begin{aligned} a_i^* &= \arg \max_{a_i \in \mathbb{A}} f((a_i, a_{i-1}^*, \dots, a_2^*, a_1^*)) \\ &\quad - f((a_{i-1}^*, \dots, a_1^*)) \quad \forall i = 1, 2, \dots, k. \end{aligned}$$

Then, we can derive bounds in the same way as Theorem 1, and the results are symmetric.

The results in Theorem 1 implies that for a string submodular function, we have $\sigma(O) \geq 0$. Otherwise, part (i) of Theorem 1 would imply that $f(G_K) \geq f(O)$, which is absurd. Moreover, if the function is backward monotone, then $\sigma(O) \leq 1$. From these, we get the following result, which is also derived in [12].

Corollary 1: Suppose that f is string submodular and backward monotone. Then,

$$f(G_K) \geq (1 - (1 - \frac{1}{K})^K) f(O) \geq (1 - e^{-1}) f(O).$$

Next, we use the notion of elemental forward curvature to generalize the diminishing-return property and we investigate the approximation bound using the elemental forward curvature.

Theorem 2: Consider a forward-monotone function f with elemental forward curvature η . Let O be an optimal solution to (1). Suppose that $f(G_i \oplus O) \geq f(O)$ for $i = 1, 2, \dots, K-1$. Then, any greedy string G_K satisfies

$$f(G_K) = f(O) \left(1 - (1 - \frac{1}{K_\eta})^K\right),$$

where $K_\eta = (1 - \eta^K)/(1 - \eta)$ if $\eta \neq 1$ and $K_\eta = K$ if $\eta = 1$.

Proof: We know that $\eta = 1$ is equivalent to the diminishing-return property. The proof for this case is given

in [1]. Now consider the case where $\eta \neq 1$. For any $M, N \in \mathbb{A}^*$ and $|M| \leq K$, there exists $a \in \mathbb{A}$ such that

$$\begin{aligned} & f(M \oplus N) - f(M) \\ &= \sum_{i=1}^{|N|} (f(M \oplus (a_1, \dots, a_i)) - f(M \oplus (a_1, \dots, a_{i-1}))) \\ &\leq \sum_{i=1}^{|N|} \eta^{i-1} (f(M \oplus a_i) - f(M)) \\ &\leq (1 + \eta + \eta^2 + \dots + \eta^{|N|-1}) (f(M \oplus a) - f(M)) \\ &= \frac{1 - \eta^{|N|}}{1 - \eta} (f(M \oplus a) - f(M)). \end{aligned}$$

Now let us consider the optimization problem (1) with length constraint K . Using the property of the greedy strategy and the assumptions, we have for $i = 1, 2, \dots, K$,

$$\begin{aligned} & f(G_i) - f(G_{i-1}) \\ &\geq \frac{1 - \eta}{1 - \eta^K} (f(G_{i-1} \oplus O) - f(G_{i-1})) \\ &\geq \frac{1 - \eta}{1 - \eta^K} (f(O) - f(G_{i-1})). \end{aligned}$$

Let $K_\eta = \frac{1 - \eta^K}{1 - \eta}$. Therefore, by recursion, we have

$$\begin{aligned} f(G_K) &\geq \frac{1}{K_\eta} f(O) + (1 - \frac{1}{K_\eta}) f(G_{K-1}) \\ &= \frac{1}{K_\eta} f(O) \sum_{i=0}^{K-1} (1 - \frac{1}{K_\eta})^i \\ &= f(O) \left(1 - (1 - \frac{1}{K_\eta})^K \right). \end{aligned}$$

Because $1 - (1 - \frac{1}{K_\eta})^K$ is decreasing as a function of $\eta \in [0, 1]$, we obtain the following result, which is reported in [1].

Corollary 2: Consider a string submodular function f . Let O be a solution to (1). Suppose that $f(G_i \oplus O) \geq f(O)$ for $i = 1, 2, \dots, K - 1$. Then, any greedy string G_K satisfies

$$f(G_K) \geq (1 - (1 - \frac{1}{K})^K) f(O) > (1 - e^{-1}) f(O).$$

Next we combine the results in Theorems 1 and 2 to get the following result.

Proposition 1: Consider a forward-monotone function f with elemental forward curvature η . Let O be a solution to (1). Then, any greedy string G_K satisfies

- (i) $f(G_K) \geq \frac{1}{\sigma(O)} \left(1 - \left(1 - \frac{\sigma(O)}{K_\eta} \right)^K \right) f(O)$.
- (ii) $f(G_K) \geq (1 - \max_{i=1, \dots, K-1} \epsilon(G_i)) \frac{K}{K_\eta} f(O)$.

The proof is given in Appendix A. We note that the condition $f(G_i \oplus O) \geq f(O)$ in Theorem 2 is essentially captured by $\sigma(O)$.

IV. NON-UNIFORM STRUCTURE

In the last section, we considered the case where I is a uniform structure. In this section, we consider the case of non-uniform structures.

We first need the following definition. Let $M = (m_1, \dots, m_{|M|})$ and $N = (n_1, \dots, n_{|N|})$ be two strings in \mathbb{A}^* . We write $M \prec N$ if there exists a sequence of strings L_i such that

$$\begin{aligned} N &= L_1 \oplus (m_1, \dots, m_{i_1}) \oplus L_2 \oplus (m_{i_1+1}, \dots, m_{i_2}) \oplus \\ &\dots \oplus (m_{i_{k-1}+1}, \dots, m_{|M|}) \oplus L_{k+1}. \end{aligned}$$

In other words, we can remove some elements in N to get M . Note that \prec is a weaker notion of dominance than \preceq defined earlier in Section II.

Now we state the definition of a non-uniform structure, analogous to the definition of independent sets in matroid theory. A subset I of \mathbb{A}^* is called a *non-uniform structure* if it satisfies the following conditions:

- 1) I is non-empty;
- 2) *Hereditary*: $\forall M \in I, N \prec M$ implies that $N \in I$;
- 3) *Augmentation*: $\forall M, N \in I$ and $|M| < |N|$, there exists an element $x \in \mathbb{A}$ in the string N such that $M \oplus (x) \in I$.

By analogy with the definition of a matroid, we call the pair (\mathbb{A}, I) a *string-matroid*. We assume that there exists K such that for all $M \in I$ we have $|M| \leq K$ and there exists a $N \in I$ such that $|N| = K$. We call such a string N a maximal string. We are interested in the following optimization problem:

$$\begin{aligned} & \text{maximize } f(N) \\ & \text{subject to } N \in I. \end{aligned} \tag{8}$$

Note that if the function is forward monotone, then the maximum of the function subject to a string-matroid constraint is achieved at a maximal string in the matroid. The greedy strategy $G_k = (a_1^*, \dots, a_k^*)$ in this case is given by

$$\begin{aligned} a_i^* &= \arg \max_{a_i \in \mathbb{A} \text{ and } (a_1^*, \dots, a_{i-1}^*, a_i) \in I} f((a_1^*, a_2^*, \dots, a_{i-1}^*, a_i)) \\ &\quad - f((a_1^*, a_2^*, \dots, a_{i-1}^*)) \quad \forall i = 1, 2, \dots, k. \end{aligned}$$

Compared with (1), at each stage i , instead of choosing a_i arbitrarily in \mathbb{A} to maximize the step-wise gain in the objective function, we also have to choose the action a_i such that the concatenated string $(a_1^*, \dots, a_{i-1}^*, a_i)$ is in the non-uniform structure I . We first establish the following theorem.

Theorem 3: For any $N \in I$, there exists a permutation of N , denoted by $\mathcal{P}(N) = (\hat{n}_1, \hat{n}_2, \dots, \hat{n}_{|N|})$, such that for $i = 1, \dots, |N|$ we have

$$f(G_{i-1} \oplus (\hat{n}_i)) - f(G_{i-1}) \leq f(G_i) - f(G_{i-1}).$$

Proof: We prove this claim by induction on $i = |N|, \dots, 1$. If $i = |N|$, consider $G_{|N|-1}$ and N , we know from Axiom 3) of string-matroid that there exists an element of N , denoted by $\hat{n}_{|N|}$ (we can always permute this element to the end of the string with a certain permutation), such that $G_{|N|-1} \oplus (\hat{n}_{|N|}) \in I$. Moreover, we know that the greedy way of selecting $a_{|N|}^*$ gives the largest gain in the objective function. Therefore, we obtain $f(G_{|N|-1} \oplus (\hat{n}_{|N|})) - f(G_{|N|-1}) \leq f(G_{|N|}) - f(G_{|N|-1})$.

Now let us assume that the claim holds for all $i > i_0$ and the corresponding elements are $\{\hat{n}_{i_0+1}, \dots, \hat{n}_{|N|}\}$. Next we show that the claim is true for $i = i_0$. Let \tilde{N}_{i_0} be the string

after we remove the elements in $\{\hat{n}_{i_0+1}, \dots, \hat{n}_{|N|}\}$ from the original string N . We know that $|G_{i_0-1}| < |\hat{N}_{i_0}|$, therefore, there exists an element from \hat{N}_{i_0} , denoted by \hat{n}_{i_0} , such that $G_{i_0-1} \oplus (\hat{n}_{i_0}) \in I$. Using the property of the greedy strategy, we obtain $f(G_{i_0-1} \oplus (\hat{n}_{i_0})) - f(G_{i_0-1}) \leq f(G_{i_0}) - f(G_{i_0-1})$. This concludes the induction proof. ■

Next we investigate the approximation bounds for the greedy strategy using the total curvatures.

Theorem 4: Let O be an optimal strategy for (8). Suppose that f is a string submodular function. Then, a greedy strategy G_K satisfies

- (i) $f(G_K) \geq \frac{1}{1+\sigma(O)}f(O)$,
- (ii) $f(G_K) \geq (1 - \epsilon(G_K))f(O)$.

Proof: (i) By the definition of the total backward curvature, we know that

$$f(G_K \oplus O) - f(O) \geq (1 - \sigma(O))f(G_K).$$

Let $O = (o_1, \dots, o_K)$. We have

$$\begin{aligned} f(O) &\leq f(G_K \oplus O) - (1 - \sigma(O))f(G_K) \\ &= f(G_K) - (1 - \sigma(O))f(G_K) + f(G_K \oplus O) - f(G_K). \end{aligned}$$

Moreover, by the diminishing-return property, we have

$$\begin{aligned} f(G_K \oplus O) - f(G_K) &= \sum_{i=1}^K (f(G_K \oplus (o_1, \dots, o_i)) - f(G_K \oplus (o_1, \dots, o_{i-1}))) \\ &\leq \sum_{i=1}^K (f(G_K \oplus (o_i)) - f(G_K)). \end{aligned}$$

By Theorem 3, we know that there exists a permutation of O : $\mathcal{P}(O) = (\hat{o}_1, \dots, \hat{o}_{|O|})$ such that

$$f(G_{i-1} \oplus (\hat{o}_i)) - f(G_{i-1}) \leq f(G_i) - f(G_{i-1}),$$

for $i = 1, \dots, K$. Therefore, by the diminishing-return property, we have

$$\begin{aligned} &\sum_{i=1}^K (f(G_K \oplus (o_i)) - f(G_K)) \\ &\leq \sum_{i=1}^K (f(G_{i-1} \oplus (\hat{o}_i)) - f(G_{i-1})) \\ &\leq \sum_{i=1}^K (f(G_i) - f(G_{i-1})) = f(G_K). \end{aligned}$$

From the above equations, we have

$$\begin{aligned} f(O) &\leq f(G_K) + f(G_K) - (1 - \sigma(O))f(G_K) \\ &= (1 + \sigma(O))f(G_K). \end{aligned}$$

We obtain the desired result.

(ii) From the definition of total forward curvature, we have

$$f(G_K \oplus O) - f(G_K) \geq (1 - \epsilon(G_K))f(O).$$

From the proof of part (i), we also know that $f(G_K \oplus O) - f(G_K) \leq f(G_K)$. Therefore, we have $f(G_K) \geq (1 - \epsilon(G_K))f(O)$. ■

The inequality in (i) above is a generalization of a result on maximizing submodular set functions with a general matroid constraint [8]. The submodular set counterpart involves total curvature, whereas the string version involves total *backward* curvature. Note that if f is backward monotone, then $\sigma(O) \leq 1$. We now state an immediate result from Theorem 4.

Corollary 3: Suppose that f is a string submodular function and f is backward monotone. Then, the greedy strategy achieves at least a $1/2$ -approximation of the optimal strategy.

Next we generalize the diminishing-return property using the elemental forward curvature.

Theorem 5: Suppose that f is a forward-monotone function with elemental forward curvature η . Suppose that $f(G_K \oplus O) \geq f(O)$. If $\eta \leq 1$, then

$$f(G_K) \geq \frac{1}{1 + \eta}f(O).$$

If $\eta > 1$, then

$$f(G_K) \geq \frac{1}{1 + \eta^{2K-1}}f(O).$$

Proof: Let $O = (o_1, \dots, o_K)$. From the definition of elemental forward curvature, we know that

$$\begin{aligned} f(G_K \oplus O) - f(G_K) &= \sum_{i=1}^K (f(G_K \oplus (o_1, \dots, o_i)) - f(G_K \oplus (o_1, \dots, o_{i-1}))) \\ &\leq \sum_{i=1}^K (f(G_{K-1} \oplus (o_i)) - f(G_{K-1}))\eta^i \\ &\leq \begin{cases} \eta \sum_{i=1}^K (f(G_{K-1} \oplus (o_i)) - f(G_{K-1})), & \text{if } \eta \leq 1 \\ \eta^K \sum_{i=1}^K (f(G_{K-1} \oplus (o_i)) - f(G_{K-1})), & \text{if } \eta > 1. \end{cases} \end{aligned}$$

From Theorem 3, we know that there exists a permutation \mathcal{P} of O : $\mathcal{P}(O) = (\hat{o}_1, \dots, \hat{o}_K)$, such that

$$f(G_{i-1} \oplus (\hat{o}_i)) - f(G_{i-1}) \leq f(G_i) - f(G_{i-1}),$$

for $i = 1, \dots, K$. Moreover, by the definition of elemental forward curvature, we have

$$\begin{aligned} &\sum_{i=1}^K (f(G_{K-1} \oplus (o_i)) - f(G_{K-1})) \\ &= \sum_{i=1}^K (f(G_{K-1} \oplus (\hat{o}_i)) - f(G_{K-1})) \\ &\leq \sum_{i=1}^K \eta^{K-i} (f(G_{i-1} \oplus (\hat{o}_i)) - f(G_{i-1})) \\ &\leq \begin{cases} \sum_{i=1}^K (f(G_{i-1} \oplus (\hat{o}_i)) - f(G_{i-1})), & \text{if } \eta \leq 1 \\ \eta^{K-1} \sum_{i=1}^K (f(G_{i-1} \oplus (\hat{o}_i)) - f(G_{i-1})), & \text{if } \eta > 1. \end{cases} \\ &\leq \begin{cases} f(G_K), & \text{if } \eta \leq 1 \\ \eta^{K-1} f(G_K), & \text{if } \eta > 1. \end{cases} \end{aligned}$$

Therefore, we have

$$f(O) \leq \begin{cases} (1 + \eta)f(G_K), & \text{if } \eta \leq 1 \\ (1 + \eta^{2K-1})f(G_K), & \text{if } \eta > 1. \end{cases}$$

■

This result is similar to that in [11]. However, the second bound in Theorem 5 is different from that in [11]. This is because the proof in [11] uses the fact that the value of a set function evaluated at a given set does not change with respect to any permutation of this set. However, the value of a string function evaluated at a given string might change with respect to a permutation of this string. Recall that the elemental forward curvature for a string submodular function is not larger than 1. We obtain the following result.

Corollary 4: Suppose that f is a string submodular function and $f(G_K \oplus O) \geq f(O)$. Then, the greedy strategy achieves at least a $1/2$ -approximation of the optimal strategy.

Now we combine the results for total and elemental curvatures to get the following.

Proposition 2: Suppose that f is a forward-monotone function with elemental forward curvature η . Then, a greedy strategy G_K satisfies

$$(i) \ f(G_K) \geq \frac{1}{\sigma(O) + \bar{\eta}} f(O),$$

$$(ii) \ f(G_K) \geq \frac{1 - \epsilon(G_K)}{\bar{\eta}} f(O),$$

where $\bar{\eta} = \eta$ if $\eta \leq 1$ and $\bar{\eta} = \eta^{2K-1}$ if $\eta > 1$.

The proof is given in Appendix B. From these results, we know that when f is string submodular, $\eta \in [0, 1]$ and we must have $\sigma(O) + \eta \geq 1$ and $\epsilon(G_K) + \eta \geq 1$. From Theorems 1, 2, 4, and 5, we know that the approximations of greedy relative to optimal get better as the total forward/backward curvature or the elemental forward curvature decreases to 0. However, the above inequalities indicate that the approximations with total curvature constraints and elemental forward curvature constraint cannot get arbitrarily good simultaneously. When equality in either case holds, the greedy strategy is optimal. A special case for this scenario is when the objective function is *string-linear*: $f(M \oplus N) = f(M) + f(N)$ for all $M, N \in \mathbb{A}^*$, i.e., $\eta = 1$ and $\sigma = \epsilon = 0$. Recall that $0 \leq \sigma(O) \leq \sigma$ and $0 \leq \epsilon(G_K) \leq \epsilon$.

Remark: The above proposition and the discussions afterward easily generalize to the framework of submodular set functions.

V. APPLICATIONS

In this section, we investigate two applications of string submodular functions with curvature constraints.

A. Strategies for Accomplishing Tasks

Consider an objective function of the following form:

$$f((a_1, \dots, a_k)) = \frac{1}{n} \sum_{i=1}^n \left(1 - \prod_{j=1}^k (1 - p_i^j(a_j)) \right).$$

We can interpret this objective function as follows. We have n subtasks, and by choosing action a_j at stage j there is a probability $p_i^j(a_j)$ of accomplishing the i th subtask. Therefore, the objective function is the expected fraction of subtasks that are accomplished after performing (a_1, \dots, a_k) . Suppose that p_i^j is independent of j for all i ; i.e., the probability of accomplishing the i th subtask by choosing an action does not depend on the stage at which the action is chosen. Then, it

is obvious that the objective function does not depend on the order of actions. In this special case, the objective function is a submodular set function and therefore the greedy strategy achieves at least a $(1 - e^{-1})$ -approximation of the optimal strategy. Moreover, it turns out that this special case is closely related to several previously studied problems, such as min-sum set cover [32], pipelined set cover [33], social network influence [34], and coverage-aware self scheduling in sensor networks [35]. In this paper, we generalize the special case to the situation where p_i^j depends on j . Applications of this generalization include advising campaign for political voting, etc. Without loss of generality, we will consider the special case where $n = 1$ (our analysis easily generalizes to arbitrary n). In this case, we have

$$f((a_1, \dots, a_k)) = 1 - \prod_{j=1}^k (1 - p^j(a_j)).$$

For each $a \in \mathbb{A}$, we assume that $p^j(a)$ takes values in $[L(a), U(a)]$, where $0 < L(a) < U(a) < 1$. Moreover, let

$$c(a) = \frac{1 - U(a)}{1 - L(a)}.$$

Obviously, $c(a) \in (0, 1)$. The forward-monotone property is easy to check: For any $M, N \in \mathbb{A}^*$, the statement $f(M \oplus N) \geq f(M)$ is obviously true.

1) *Uniform Structure:* We first consider the maximization problem under the uniform structure constraint. The elemental forward curvature in this case is

$$\eta = \max_{a_i, a_j} \frac{(1 - p^i(a_i))p^j(a_j)}{p^i(a_j)}.$$

Suppose that $\hat{U} = \max_{a \in \mathbb{A}} U(a)$ and $\hat{L} = \min_{a \in \mathbb{A}} L(a)$. Then, we have

$$\eta \leq \frac{(1 - \hat{L})\hat{U}}{\hat{L}},$$

for all possible combinations of probability values p^j , $j = 1, \dots$. Note that the function is submodular if and only if $\eta \leq 1$. From the above equation, we conclude that f is submodular if

$$\frac{(1 - \hat{L})\hat{U}}{\hat{L}} \leq 1.$$

Therefore, a sufficient condition for string submodularity is

$$\hat{L}^{-1} - \hat{U}^{-1} \leq 1.$$

In Theorem 1, instead of calculating the total backward curvature with respect to the optimal strategy, we calculate the total backward curvature for $K \leq |M| < 2K$:

$$\hat{\sigma} = \max_{a \in \mathbb{A}, K \leq |M| < 2K} \left\{ 1 - \frac{f((a) \oplus M) - f(M)}{f((a)) - f(\emptyset)} \right\} \quad (9)$$

$$= 1 - \min_{a \in \mathbb{A}, K \leq |M| < 2K} \left\{ \frac{f((a) \oplus M) - f(M)}{f((a)) - f(\emptyset)} \right\}. \quad (10)$$

We have

$$\begin{aligned} & \frac{f((a) \oplus M) - f(M)}{f((a)) - f(\emptyset)} \\ &= \frac{\prod_{j=1}^{|M|} (1 - p^j(a_j)) - (1 - p^1(a)) \prod_{j=1}^{|M|} (1 - p^{j+1}(a_j))}{p^1(a)}. \end{aligned}$$

We then provide an upper bound for the total backward curvature for all possible combination of p^j . The minimum of the above term is achieved at $p^j(a_j) = \hat{U}$ and $p^{j+1}(a_j) = \hat{L}$:

$$\begin{aligned} & \min_{a \in \mathbb{A}, K \leq |M| < 2K} \left\{ \frac{f((a) \oplus M) - f(M)}{f((a)) - f(\emptyset)} \right\} \\ & \geq \min_{a \in \mathbb{A}, K \leq k < 2K} \frac{(1 - \hat{U})^k - (1 - p^1(a))(1 - \hat{L})^k}{p^1(a)} \\ & \geq \min_{K \leq k < 2K} \frac{(1 - \hat{U})^k - (1 - \hat{L})^{k+1}}{\hat{L}}. \end{aligned}$$

From this we can derive an upper bound for the total backward curvature and use the upper bound in Theorem 1. For example, suppose that $(1 - \hat{U})^k / (1 - \hat{L})^k \geq 1 - \hat{L}$ for all $K \leq k < 2K$. Then, we achieve the minimum at $2K - 1$:

$$\begin{aligned} & \min_{K \leq k < 2K} \frac{(1 - \hat{U})^k - (1 - \hat{L})^{k+1}}{\hat{L}} \\ & = \frac{(1 - \hat{U})^{2K-1} - (1 - \hat{L})^{2K}}{\hat{L}}. \end{aligned}$$

Moreover, it is easy to verify that $\sigma(O) \leq \hat{\sigma}$. We can therefore use $\hat{\sigma}$ to derive a lower bound for the approximation of the greedy strategy.

Instead of calculating the total forward curvature with respect to the greedy strategy G_i , we calculate

$$\hat{\epsilon}_i = \max_{a \in \mathbb{A}, i \leq |M| < i+K} \left\{ 1 - \frac{f(M \oplus (a)) - f(M)}{f((a)) - f(\emptyset)} \right\} \quad (11)$$

$$= 1 - \min_{a \in \mathbb{A}, i \leq |M| < i+K} \left\{ \frac{f(M \oplus (a)) - f(M)}{f((a)) - f(\emptyset)} \right\} \quad (12)$$

$$= 1 - \min_{a \in \mathbb{A}, i \leq |M| < i+K} \frac{\prod_{j=1}^{|M|} (1 - p^j(a_j)) p^1(a)}{p^1(a)} \quad (13)$$

$$\geq 1 - (1 - \hat{U})^{i+K-1}. \quad (14)$$

It is easy to show that $\epsilon(G_i) \leq \hat{\epsilon}_i$. Moreover, we have

$$\max_{i=1, \dots, K-1} \epsilon(G_i) \leq \max_{i=1, \dots, K-1} \hat{\epsilon}_i \leq 1 - (1 - \hat{U})^{2K-2}.$$

We can substitute this term in Theorem 1 and get a lower bound for the approximation of the optimal strategy that the greedy strategy is guaranteed to achieve.

In Theorem 2, we need the additional assumption that $f(G_i \oplus O) \leq f(O)$ for $i = 1, \dots, K - 1$, which can be written as

$$\prod_{j=1}^K (1 - p^j(o_j)) \geq \prod_{t=1}^i (1 - p^j(a_j^*)) \prod_{j=1}^K (1 - p^{j+i}(o_j)). \quad (15)$$

We know that

$$\prod_{j=1}^K (1 - p^j(o_j)) \geq \prod_{j=1}^K (1 - U(o_j))$$

and

$$\prod_{t=1}^i (1 - p^j(a_j^*)) \prod_{j=1}^K (1 - p^{j+i}(o_j)) \leq \prod_{j=1}^K (1 - L(o_j)) (1 - p^1(a_1^*)).$$

Therefore, a sufficient condition for (15) is

$$1 - p^1(a_1^*) \leq \frac{\prod_{j=1}^K (1 - U(o_j))}{\prod_{j=1}^K (1 - L(o_j))} = \prod_{j=1}^K c(o_j).$$

Let $c = \min_{a \in \mathbb{A}} c(a)$. Suppose that we have

$$p^1(a_1^*) \geq 1 - c^K.$$

Then, $f(G_i \oplus O) \leq f(O)$ for $i = 1, \dots, K - 1$.

2) *Non-uniform Structure*: The calculation for the case of non-uniform structure uses a similar analysis. For example, in Theorem 4, the calculation of the total backward curvature can be calculated in the same way as the case of uniform structure.

Now let us consider the backward monotone property required in Theorem 5: $f(G_K \oplus O) \geq f(O)$. This condition is much weaker than that in Theorem 2, and can be rewritten as

$$\prod_{j=1}^K (1 - p^j(o_j)) \geq \prod_{t=1}^K (1 - p^j(a_j^*)) \prod_{j=1}^K (1 - p^{j+K}(o_j)).$$

A sufficient condition for the above inequality is $1 - \hat{U} \geq (1 - \hat{L})^2$. Recall that the function is string submodular if

$$\eta \leq \frac{(1 - \hat{L})\hat{U}}{\hat{L}} \leq 1.$$

Combining the above two inequalities, we have

$$\begin{aligned} \eta & \leq \frac{(1 - \hat{L})\hat{U}}{\hat{L}} \leq \frac{(1 - \hat{L})(1 - (1 - \hat{L})^2)}{\hat{L}} \\ & = (1 - \hat{L})(2 - \hat{L}) \leq 1. \end{aligned}$$

Therefore, we obtain

$$\hat{L} \geq 1 - \frac{1}{\alpha} \text{ and } \hat{U} \leq \frac{1}{\alpha},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ is the *golden ratio*.

Now let us consider the special case where $p^j(a)$ is non-increasing over j for each $a \in \mathbb{A}$. It is easy to show that the function is string submodular. Moreover, the elemental forward curvature is

$$\begin{aligned} \eta & = \max_{a_i, a_j} \frac{(1 - p^i(a_i)) p^j(a_j)}{p^i(a_j)} \\ & \leq \max_{a_i} (1 - p^i(a_i)) \\ & \leq 1 - \hat{L}. \end{aligned}$$

Therefore, using this elemental forward curvature bound, we can provide a better approximation than $(1 - e^{-1})$ for the greedy strategy.

Consider the special case where $p^j(a)$ is non-decreasing over j for each $a \in \mathbb{A}$. In this case, we have

$$\begin{aligned} \sigma(O) & \leq \hat{\sigma} = 1 - \min_{a \in \mathbb{A}, K \leq |M| < 2K} \left\{ \frac{f((a) \oplus M) - f(M)}{f((a)) - f(\emptyset)} \right\} \\ & \leq 1 - \prod_{j=1}^{|M|} (1 - p^j(a_j)) \\ & \leq 1 - (1 - \hat{U})^{2K-1}. \end{aligned}$$

Therefore, we can provide a better approximation than $(1 - e^{-1})$ for the greedy strategy.

B. Maximizing the Information Gain

In this part, we present an application of string submodular function on Bayesian estimation. Consider a signal of interest x , which takes value on \mathbb{R}^N with normal prior distribution $\mathcal{N}(\mu, P_0)$. In our example, we assume that the dimension $N = 2$. However, our analysis easily generalizes to dimensions larger than 2. Let \mathbb{D} denotes all the diagonal positive-semidefinite 2×2 matrices with power (trace) constraint:

$$\mathbb{D} = \{\text{Diag}(\sqrt{e}, \sqrt{1-e}) : e \in [0, 1]\}.$$

At each stage i , we choose a measurement matrix $A_i \in \mathbb{D}$ to get an observation y_i , which is corrupted with additive zero-mean Gaussian noise $\omega_i \sim \mathcal{N}(0, R_{\omega\omega}^i)$:

$$y_i = A_i x + \omega_i.$$

Let the posterior distribution of x given (y_1, y_2, \dots, y_k) be $\mathcal{N}(x_k, P_k)$. Moreover, the recursion for P_k is given by

$$\begin{aligned} P_k^{-1} &= P_{k-1}^{-1} + A_k^T (R_{\omega\omega}^k)^{-1} A_k \\ &= P_0^{-1} + \sum_{i=1}^k A_i^T (R_{\omega\omega}^i)^{-1} A_i. \end{aligned}$$

The entropy of the posterior distribution of x given (y_1, y_2, \dots, y_k) is $H_k = \frac{1}{2} \log \det P_k + \log(2\pi e)$. We define the information gain given (A_1, A_2, \dots, A_k) by

$$\begin{aligned} f((A_1, A_2, \dots, A_k)) &= H_0 - H_k \\ &= \frac{1}{2} (\log \det P_0 - \log \det P_k). \end{aligned}$$

The objective is to choose a string of measurement matrices subject to a length constraint K such that the information gain is maximized.

The optimality of greedy strategy and measurement matrix design problem are considered in [37] and [38], respectively. Suppose that the additive noises are independent and identically distributed. Then, it is easy to see that $f((A_1, A_2, \dots, A_k)) = f(\mathcal{P}(A_1, A_2, \dots, A_k))$ for all permutation \mathcal{P} . Moreover, the information gain is a submodular set function and $f(\emptyset) = 0$ [36]. Therefore, the greedy strategy achieves at least a $(1 - e^{-1})$ -approximation of the optimal strategy.

Consider the situation where the additive noises are independent but *not* identically distributed. Moreover, let us assume that $R_{\omega\omega}^i = \sigma_i^2 \mathcal{I}$, where \mathcal{I} denotes the identity matrix. In other words, the noise at each stage is white but the variances σ_i depend on i . The forward-monotone property is easy to see: We always gain by adding extra (noisy) measurements.

Now we investigate the sensitivity of string submodularity with respect to the varying noise variances. We claim that the function is string submodular if and only if σ_i is monotone non-decreasing with respect i . The sufficiency part is easy to understand: Because the measurement y_i becomes more noisy as i increases, the information gain at a later stage certainly cannot be larger than the information gain at an earlier stage. We show the necessity part by contradiction. Suppose that the function is string submodular and there exist k such that $\sigma_k \geq \sigma_{k+1}$. Suppose that the posterior covariance at stage

$k-1$ is $\text{Diag}(s_{k-1}, t_{k-1})$ and we choose $A_k = \text{Diag}(1, 0)$, $A_{k+1} = \text{Diag}(0, 1)$. We have

$$\begin{aligned} f(A_k \oplus A_{k+1}) - f(A_k) &= \log(1 + t_k \sigma_{k+1}^{-2}) \\ &= \log(1 + t_{k-1} \sigma_{k+1}^{-2}) \geq \log(1 + t_{k-1} \sigma_k^{-2}) \\ &= f(A_{k+1}) - f(\emptyset). \end{aligned}$$

This contradicts with the diminishing-return property and finishes the argument. In fact, it is easy to show that $\eta \leq 1$ if and only if the sequence of noise variance is non-decreasing. In this case, the greedy strategy achieves at least a factor (better than $(1 - e^{-1})$) of the optimal strategy.

For general cases where the noise variance sequence is not necessarily non-decreasing. We will provide upper and lower bounds for the elemental forward curvature. The lower bound of the elemental forward curvature characterizes the best bound for the greedy strategy we can possibly obtain. The upper bound of the elemental forward curvature characterizes the worst bound for the greedy strategy we can possibly obtain. Instead of considering the forward elemental curvature directly, it suffices to calculate the curvature over all M such that $|M| \leq 2K - 2$:

$$\hat{\eta} = \max_{A_i, A_j \in I, |M| \leq 2K-2} \frac{f(M \oplus (A_i) \oplus (A_j)) - f(M \oplus (A_i))}{f(M \oplus (A_j)) - f(M)}. \quad (16)$$

It is easy to see that $\hat{\eta} \leq \eta$ and we can use $\hat{\eta}$ to improve the results in Theorems 2 and 5.

For simplicity, let $P_0 = \text{Diag}(s_0, t_0)$. Without loss of generality, we assume that $s_0 \geq t_0$. Let $M = (A_1, A_2, \dots, A_{|M|})$ where $A_k = \text{Diag}(\sqrt{e_k}, \sqrt{1-e_k})$ for $k = 1, \dots, |M|$. Let $P_{|M|} = \text{Diag}(s_{|M|}, t_{|M|})$ where

$$\begin{aligned} s_0^{-1} &\leq s_{|M|}^{-1} = s_0^{-1} + \sum_{i=1}^{|M|} \sigma_i^{-2} e_i \leq s_0^{-1} + \sum_{i=1}^{|M|} \sigma_i^{-2}, \\ t_0^{-1} &\leq t_{|M|}^{-1} = t_0^{-1} + \sum_{i=1}^{|M|} \sigma_i^{-2} (1 - e_i) \leq t_0^{-1} + \sum_{i=1}^{|M|} \sigma_i^{-2}, \end{aligned}$$

and $s_{|M|}^{-1} + t_{|M|}^{-1} = s_0^{-1} + t_0^{-1} + \sum_{i=1}^{|M|} \sigma_i^{-2}$. Consider the special case where $A_i = \text{Diag}(1, 0)$ and $A_j = \text{Diag}(0, 1)$. The denominator of (16) can be written as

$$f(M \oplus (A_j)) - f(M) = \log(1 + t_{|M|} \sigma_{|M|+1}^{-2}).$$

The numerator in (16) can be written as

$$\begin{aligned} f(M \oplus (A_i) \oplus (A_j)) - f(M \oplus (A_i)) \\ = \log(1 + t_{|M|+1} \sigma_{|M|+2}^{-2}). \end{aligned}$$

Therefore, we have

$$\hat{\eta} \geq \max_{|M| \leq 2K-2} \frac{\log(1 + t_{|M|+1} \sigma_{|M|+2}^{-2})}{\log(1 + t_{|M|} \sigma_{|M|+1}^{-2})}. \quad (17)$$

We know that $t_{|M|+1} = t_{|M|} \geq (t_0^{-1} + \sum_{i=1}^{|M|} \sigma_i^{-2})^{-1}$, where $=$ is achieved when $e_k = 0$ for $k = 1, \dots, |M|$. Let

$$k = \arg \max_{|M|=0, \dots, 2K-2} \frac{\sigma_{|M|+2}^{-2}}{\sigma_{|M|+1}^{-2}}.$$

If $\sigma_{k+2}^{-2}/\sigma_{k+1}^{-2} \leq 1$, then (17) is a monotone increasing function of $t_{|M|}$. We obtain

$$\hat{\eta} \geq \frac{\log(1 + t_0 \sigma_{k+2}^{-2})}{\log(1 + t_0 \sigma_{k+1}^{-2})}.$$

If $\sigma_{k+2}^{-2}/\sigma_{k+1}^{-2} > 1$, then (17) is a monotone decreasing function of $t_{|M|}$. We obtain

$$\hat{\eta} \geq \frac{\log(1 + (t_0^{-1} + \sum_{i=1}^k \sigma_i^{-2})^{-1} \sigma_{k+2}^{-2})}{\log(1 + (t_0^{-1} + \sum_{i=1}^k \sigma_i^{-2})^{-1} \sigma_{k+1}^{-2})}.$$

Now let us further assume that σ_i randomly takes a value from $[a, b]$, where $0 < a < b$. We wish to provide a lower bound for $\hat{\eta}$ for all possible realization of noise variances. From (17), we obtain

$$\hat{\eta} \geq \max_{|M| \leq 2K-2} \frac{\log(1 + t_{|M|} b^{-2})}{\log(1 + t_{|M|} a^{-2})}.$$

Note that the above lower bound is a monotone increasing function of $t_{|M|}$. We obtain

$$\hat{\eta} \geq \frac{\log(1 + t_0 b^{-2})}{\log(1 + t_0 a^{-2})}.$$

Next we derive the upper bound for $\hat{\eta}$. We first derive an upper bound for the numerator in (16), which is given by (18) on the next page.

We then derive the lower bound of the denominator in (16). It is easy to show that the minimum is achieved at the boundary:

$$\begin{aligned} f(M \oplus (A_j)) - f(M) &\geq \\ \min(\log(1 + t_{|M|} \sigma_{|M|+1}^{-2}), \log(1 + s_{|M|} \sigma_{|M|+1}^{-2})) & \\ \geq \log(1 + \min(s_{|M|} \sigma_{|M|+1}^{-2}, t_{|M|} \sigma_{|M|+1}^{-2})) & \\ \geq \log(1 + (t_0^{-1} + \sum_{i=1}^{2K-2} \sigma_i^{-2})^{-1} \min_{i=1, \dots, 2K} \sigma_i^{-2}). & \end{aligned}$$

Therefore, we can derive an upper bound for the curvature as follows:

$$\hat{\eta} \leq \frac{\log \frac{1}{4} (1 + s_0 t_0^{-1} + s_0 \sum_{i=1}^{2K} \sigma_i^{-2}) (1 + \frac{s_0^{-1} + \max_{i=1, \dots, 2K} \sigma_i^{-2}}{(t_0^{-1} + \sigma_1^{-2})})}{\log(1 + (t_0^{-1} + \sum_{i=1}^{2K-2} \sigma_i^{-2})^{-1} \min_{i=1, \dots, 2K} \sigma_i^{-2})}.$$

Using this upper bound, we can provide an approximation bound for the greedy strategy. We note that this upper bound is not extremely tight in the sense that it does not increase significantly with K only if s_0 or σ_i^{-2} are sufficiently small.

Now consider the case where σ_i randomly takes a value from $[a, b]$, where $0 < a < b$. We wish to provide an upper bound for $\hat{\eta}$ for all possible realization of noise variances. Therefore, we have

$$\hat{\eta} \leq \frac{\log \frac{1}{4} (1 + s_0 t_0^{-1} + 2s_0 K a^{-2}) (1 + \frac{s_0^{-1} + a^{-2}}{(t_0^{-1} + b^{-2})})}{\log(1 + t_0 (1 + t_0 (2K - 2) a^{-2})^{-1} b^{-2})}.$$

For Theorem 2, we need to have $f(G_i \oplus O) \geq f(O)$ for $i = 1, 2, \dots, K - 1$. Let $a^* \in \mathbb{D}$ be a greedy action. We will provide a sufficient condition such that $f((a^*) \oplus M) \geq f(M)$

for all M with length K . Suppose that $\sigma_i \in [a, b]$ for all i . Let $a^* = \text{Diag}(\sqrt{e^*}, \sqrt{1 - e^*})$ and $M = (a_1, \dots, a_K)$, where $a_t = \text{Diag}(\sqrt{e_t}, \sqrt{1 - e_t})$ for all t . The inequality we need to verify can be written as

$$\begin{aligned} &\log(1 + s_0 (\sigma_1^{-2} e^* + \sum_{t=1}^K \sigma_{t+1}^{-2} e_t)) \times \\ &(1 + t_0 (\sigma_1^{-2} (1 - e^*) + \sum_{t=1}^K \sigma_{t+1}^{-2} (1 - e_t))) \\ &\geq \log(1 + s_0 (\sum_{t=1}^K \sigma_t^{-2} e_t)) (1 + t_0 (\sum_{t=1}^K \sigma_t^{-2} (1 - e_t))). \end{aligned}$$

We first calculate the value of e^* . It is easy to show that the objective function after applying (a^*) achieves the maximum when

$$e^* = \frac{1 + \frac{t_0^{-1} - s_0^{-1}}{\sigma_1^{-1}}}{2}.$$

Because e^* can only take values in $[0, 1]$, in the case where $(t_0^{-1} - s_0^{-1})/\sigma_1^{-1} \geq 1$, the maximum is achieved at $e^* = 1$. We will consider this case and the analysis for the case where $(t_0^{-1} - s_0^{-1})/\sigma_1^{-1} < 1$ is similar and omitted. Therefore, to show the above inequality, it suffices to show

$$\begin{aligned} &\log(1 + s_0 \sigma_1^{-2} + (s_0 \sum_{t=1}^K \sigma_{t+1}^{-2} e_t)) (1 + t_0 (\sum_{t=1}^K \sigma_{t+1}^{-2} (1 - e_t))) \\ &\geq \log(1 + s_0 (\sum_{t=1}^K \sigma_t^{-2} e_t)) (1 + t_0 (\sum_{t=1}^K \sigma_t^{-2} (1 - e_t))). \end{aligned}$$

Removing the log on both sides of the inequality, we obtain

$$\begin{aligned} &(1 + s_0 \sum_{t=1}^K \sigma_{t+1}^{-2} e_t) (1 + t_0 \sum_{t=1}^K \sigma_{t+1}^{-2} (1 - e_t)) \\ &+ s_0 \sigma_1^{-2} (1 + t_0 \sum_{t=1}^K \sigma_{t+1}^{-2} (1 - e_t)) \\ &\geq (1 + s_0 \sum_{t=1}^K \sigma_t^{-2} e_t) (1 + t_0 \sum_{t=1}^K \sigma_t^{-2} (1 - e_t)). \end{aligned}$$

Rearranging terms, we obtain (19), where $\mathbb{I}_t = 1$ if $\sigma_{t+1}^{-2} \leq \sigma_t^{-2}$ and $\mathbb{I}_t = 0$ if $\sigma_{t+1}^{-2} > \sigma_t^{-2}$.

From this we obtain a sufficient condition for $f((a^*) \oplus M) \geq f(M)$ to hold

$$\frac{b^{-2}}{a^{-2} - b^{-2}} \geq \frac{K^2}{4} t_0 (a^{-2} + b^{-2}) + 1.$$

We have shown before that the elemental curvature is not larger than 1 if and only if the noise variance is non-decreasing. Moreover, if the above inequality holds, which requires that either the length of the variance interval $[a, b]$ or K is sufficiently small, then we can get the $(1 - e^{-1})$ bound.

VI. CONCLUSION

In this paper, we have introduced the notion of total forward/backward and elemental forward curvature for functions defined on strings. We have shown that the greedy strategy

$$\begin{aligned}
& f(M \oplus (A_i) \oplus (A_j)) - f(M \oplus (A_i)) = \log(1 + s_{|M|+1} \sigma_{|M|+2}^{-2} e_j)(1 + t_{|M|+1} \sigma_{|M|+2}^{-2} (1 - e_j)) \\
& = \log(s_{|M|+1}^{-1} + \sigma_{|M|+2}^{-2} e_j)(t_{|M|+1}^{-1} + \sigma_{|M|+2}^{-2} (1 - e_j)) + \log s_{|M|+1} t_{|M|+1} \\
& \leq \log \left(\frac{s_{|M|+1}^{-1} + \sigma_{|M|+2}^{-2} e_j + t_{|M|+1}^{-1} + \sigma_{|M|+2}^{-2} (1 - e_j)}{2} \right)^2 + \max(-\log(s_0^{-1} + \sum_{i=1}^{|M|+1} \sigma_i^{-2}) t_0^{-1}, -\log s_0^{-1} (t_0^{-1} + \sum_{i=1}^{|M|+1} \sigma_i^{-2})) \\
& = \log \left(\frac{s_0^{-1} + t_0^{-1} + \sum_{i=1}^{|M|+2} \sigma_i^{-2}}{2} \right)^2 - \log s_0^{-1} (t_0^{-1} + \sum_{i=1}^{|M|+1} \sigma_i^{-2}) \\
& = \log \left(\frac{1 + s_0 t_0^{-1} + s_0 \sum_{i=1}^{|M|+2} \sigma_i^{-2}}{2} \right) + \log \left(\frac{s_0^{-1} + t_0^{-1} + \sum_{i=1}^{|M|+2} \sigma_i^{-2}}{2(t_0^{-1} + \sum_{i=1}^{|M|+1} \sigma_i^{-2})} \right) \\
& \leq \log \left(\frac{1 + s_0 t_0^{-1} + s_0 \sum_{i=1}^{2K} \sigma_i^{-2}}{2} \right) + \log \left(\frac{1}{2} \left(1 + \frac{s_0^{-1} + \max_{i=1, \dots, 2K} \sigma_i^{-2}}{(t_0^{-1} + \sigma_1^{-2})} \right) \right).
\end{aligned} \tag{18}$$

$$\begin{aligned}
& s_0 \sum_{t=1}^K e_t (\sigma_{t+1}^{-2} - \sigma_t^{-2}) + t_0 \sum_{t=1}^K (1 - e_t) (\sigma_{t+1}^{-2} - \sigma_t^{-2}) + s_0 \sigma_1^{-2} (1 + t_0 \sum_{t=1}^K \sigma_{t+1}^{-2} (1 - e_t)) \\
& + s_0 t_0 \left(\sum_{t=1}^K \sigma_{t+1}^{-2} e_t \right) \left(\sum_{t=1}^K \sigma_{t+1}^{-2} (1 - e_t) \right) - s_0 t_0 \left(\sum_{t=1}^K \sigma_t^{-2} e_t \right) \left(\sum_{t=1}^K \sigma_t^{-2} (1 - e_t) \right) \\
& \geq s_0 \sum_{t=1}^K (\sigma_{t+1}^{-2} - \sigma_t^{-2}) \mathbb{I}_t + t_0 \sum_{t=1}^K (\sigma_{t+1}^{-2} - \sigma_t^{-2}) (1 - \mathbb{I}_t) \\
& + s_0 \sigma_1^{-2} (1 + t_0 \sum_{t=1}^K \sigma_{t+1}^{-2} (1 - e_t)) + s_0 t_0 (b^{-4} - a^{-4}) \left(\sum_{t=1}^K e_t \right) \left(\sum_{t=1}^K (1 - e_t) \right) \\
& \geq s_0 (b^{-2} - a^{-2}) + s_0 b^{-2} + \frac{K^2}{4} s_0 t_0 (b^{-4} - a^{-4}).
\end{aligned} \tag{19}$$

achieves a good approximation of the optimal strategy with constraints on these curvatures. Our results contribute significantly to our understanding of the underlying algebraic structure of string submodular functions. Moreover, we have investigated two applications of string submodular functions with curvature constraints.

APPENDIX A PROOF OF PROPOSITION 1

(i) We know that $\eta = 1$ is equivalent to the diminishing-return property. The proof for this case is the same as that of Theorem 1. Now consider the case where $\eta \neq 1$. For any $M, N \in \mathbb{A}^*$ and $|M| \leq K$, there exists $a \in \mathbb{A}$ such that

$$\begin{aligned}
& f(M \oplus N) - f(M) \\
& = \frac{1 - \eta^{|N|}}{1 - \eta} (f(M \oplus a) - f(M)).
\end{aligned}$$

Now let us consider the optimization problem (1) with length constraint K . Using the property of the greedy strategy and the monotone property, we have

$$\begin{aligned}
f(G_i) - f(G_{i-1}) & \geq \frac{1 - \eta}{1 - \eta^K} (f(G_{i-1} \oplus O) - f(G_{i-1})) \\
& \geq \frac{1 - \eta}{1 - \eta^K} (f(O) - \sigma(O) f(G_{i-1})).
\end{aligned}$$

Let $K_\eta = \frac{1 - \eta^K}{1 - \eta}$. Therefore, by recursion, we have

$$\begin{aligned}
f(G_K) & \geq \frac{1}{K_\eta} f(O) + (1 - \frac{\sigma(O)}{K_\eta}) f(G_{K-1}) \\
& = \frac{1}{K_\eta} f(O) \sum_{i=0}^{K-1} (1 - \frac{\sigma(O)}{K_\eta})^i \\
& = \frac{1}{\sigma(O)} \left(1 - (1 - \frac{\sigma(O)}{K_\eta})^K \right) f(O).
\end{aligned}$$

(ii) Using a similar argument as part (i), we have

$$\begin{aligned}
f(G_i) - f(G_{i-1}) & \geq \frac{1 - \eta}{1 - \eta^K} (f(G_{i-1} \oplus O) - f(G_{i-1})) \\
& \geq \frac{1 - \eta}{1 - \eta^K} (f(G_{i-1}) - f(G_{i-1}) + (1 - \epsilon(G_{i-1})) f(O)).
\end{aligned}$$

Therefore, by recursion, we have

$$\begin{aligned}
f(G_K) & = \sum_{i=1}^K (f(G_i) - f(G_{i-1})) \\
& \geq \sum_{i=1}^K \frac{1}{K_\eta} (1 - \epsilon(G_{i-1})) f(O) \\
& \geq \frac{K_\eta}{K} (1 - \max_{i=1, \dots, K-1} \epsilon(G_i)) f(O).
\end{aligned}$$

APPENDIX B

PROOF OF PROPOSITION 2

(i) Using the definition of total backward curvature, we have

$$f(G_K \oplus O) - f(O) \geq (1 - \sigma(O))f(G_K),$$

which implies that

$$f(G_K \oplus O) - f(G_K) \geq f(O) - \sigma(O)f(G_K).$$

Using a similar argument as that of Theorem 5, we know that

$$f(G_K \oplus O) - f(G_K) \leq \bar{\eta}f(G_K).$$

Therefore, we have

$$f(G_K) \geq \frac{1}{\bar{\eta} + \sigma(O)}f(O).$$

(ii) Using the definition of total forward curvature, we have

$$f(G_K \oplus O) - f(G_K) \geq (1 - \epsilon(G_K))f(O).$$

Using a similar argument as that of Theorem 5, we know that $f(G_K \oplus O) - f(G_K) \leq \bar{\eta}f(G_K)$. Therefore, we have

$$f(G_K) \geq \frac{1 - \epsilon(G_K)}{\bar{\eta}}f(O).$$

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